Open Problems - Answers

Open Problem 2.15. An upper bound of $O(n^{1+\varepsilon})$, $\varepsilon > 0$, follows from a general simulation result contained in O. Giel: Branching program size is almost linear in formula size. Journal of Computer and System Sciences 63, 222–235, 2001.

Open Problem 3.8. The answer "yes". The solution will appear in Section 7.1.4 of "The Art of Computer Programms Vol. 4" by Donald E. Knuth. It's mainly based on the reduction algorithm leading to Theorem 3.3.4.

Open Problem 4.12. The anser is "yes", see the following paper: Bollig, B. (2008). On the OBDD complexity of the most significant bit of integer multiplication, Proc. of TAMC 2008, LNCS 4978, 306-317.

Open Problem 5.18. The answer is "yes", see B. Bollig and I. Wegener: Asymptotically optimal bounds for OBDDs and the solution of some basic OBDD problems. Journal of Computer and System Sciences 61, 558–579, 2000.

Open Problem 6.17. There are two results on the complexity of EAR_n:

– The k-OBDD size is $2^{\Omega(n^{1/2}/k)}$. (Diploma Thesis of Martin Sauerhoff. Univ. Dortmund.)

– The FBDD size is $n^{\Theta(\log n)}$. (J. Kará and D. Král: Optimal free binary decision diagrams for computation of EAR_n. 27th MFCS, LNCS 2420, 411–422, 2002.)

Open Problem 8.5. The ZBDD size of MUX_n equals $\Theta(n^2/\log n)$. B. Bollig and I. Wegener. Asymptotically optimal bounds for OBDDs and the solution of some basic OBDD problems. Journal of Computer and System Sciences 61, 558–579, 2000.

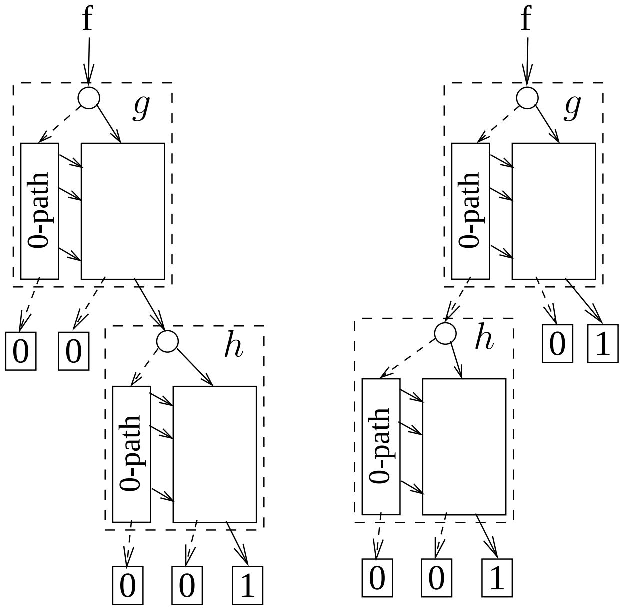
Open Problem 8.14. The answer is $n+2$. The result has been obtained by M. Sauerhoff, Univ. Dortmund.

Proposition: *Let $f\colon \{0,1\}^n \to \{0,1\}$ be representable by a read-once formula over the basis $\{\oplus, \wedge\}$ without negations. Then the OFDD size of f is $n+2$.*

Proof. This can be proven by looking closely at the construction in the proof of Theorem 8.2.9 in the book and using the following observation: For an arbitrary function φ representable by a read-once formula over $\{\oplus, \wedge\}$ without negations, it holds that $\varphi(0,\ldots,0) = 0$.

For the convenience of the reader, we repeat the proof of Theorem 8.2.9 for our special case. Let f be as described in the proposition. We construct an OFDD for f with an *isolated* 0*-path*. An OFDD has an isolated 0-path if all nodes on the path activated by the all-zeros input have indegree 1 (it is not really required that all variables are tested on this path).

First, let $f = g \wedge h$ where g and h depend on disjoint sets of variables. The top part of the OFDD for f is an OFDD with isolated 0-path for g. Since $g(0, \ldots, 0) = 0$ by our above remark, this isolated 0-path directly leads to the 0-sink (which is different from the book, where the case $g(0, \ldots, 0) = 1$ is shown). The 1-sink of the OFDD for g is identified with the source of an OFDD with isolated 0-path for h. (See Fig.1a.)



a) $f = g \wedge h$ b) $f = g \oplus h$

Figure 1.

If $f = g \oplus h$, the OFDD for f is constructed as shown in Fig. 1b. The top part is an OFDD for g with isolated 0-path. At the end of this 0-path, an OFDD for h with isolated 0-path is attached. Taking into account that $g(0, \ldots, 0) = 0$, one easily verifies that the whole OFDD computes the right function (in the case $g(0, \ldots, 0) = 1$, we would have to use an OFDD for $\overline{h}$ in the bottom part).

For any suitable function φ, let size(φ) denote the number of interior nodes of an OFDD with isolated 0-path obtained by our construction. By considering both above cases, we obviously have

$$\text{size}(f) \leq \text{size}(g) + \text{size}(h).$$

Furthermore, if the formula for f only consists of a single variable, size$(f) = 1$. It follows that the number of interior nodes in an OFDD with isolated 0-path for f is at most n. Thus the size of an OFDD as well as the size of an OFDD with isolated 0-path for f is exactly $n + 2$. $\square$

It remains open whether there are read-once formulas with larger than linear OFDD size. We conjecture that the OFDD size of the following modified version of the alternating tree function is $\Omega(n \log n)$, i. e., asymptotically matches the upper bound from Theorem 8.2.9.

Definition: Let $n = 2^k$, $k \geq 0$ even. The function $\mathrm{AT}^*_n \colon \{0,1\}^n \to \{0,1\}$ is defined by $\mathrm{AT}^*_1(x) = x$ and

$$\mathrm{AT}^*_n(u,v,x,y) := \Big(\mathrm{AT}^*_{n/4}(u) \oplus \mathrm{AT}^*_{n/4}(v)\Big) + \Big(\mathrm{AT}^*_{n/4}(x) \oplus \mathrm{AT}^*_{n/4}(y)\Big),$$

for $n \geq 4$, where u, v, x, y are disjoint variable vectors of length $n/4$ each.

Open Problem 10.4. The answer is no, see M. Sauerhoff: Computing with restricted nondeterminism: The dependence of the OBDD size on the number of nondeterministic variables. Proc. of 19.Foundation of Software Technology and Theoretical Computer Science. LNCS 1738, 342-355, 1999.

Open Problem 11.19 The following function g_n is defined on $n \times n$ boolean matrices (n even). It computes 1 iff at least one row has an odd number of ones and all columns have an odd number of ones. This function is contained in RP-OBDD and in coRP-OBDD but not in P-OBDD. This implies by Theorem 11.3.6 that the function is not in ZPP-OBDD and therefore

$$\text{ZPP-OBDD} \neq \text{RP-OBDD} \cap \text{coRP-OBDD}.$$

This result is due to Stasys Jukna: A note on the P versus NP intersected co-NP question in communication complexity. Techn. Report 2005.

Open Problem 11.21. The problem has been implicitly solved by Jürgen Forster [1] together with a known result of Martin Sauerhoff [3]. (Forster, Krause, Lokam, Mubarakzjanov, Schmitt, and Simon have recently obtained a similar result [2].)

Forster [1] has shown that probabilistic communication protocols for the inner product function IP_n with unbounded error require length $n/2$. Substituting this into a result from [3], one immediately gets an explicitly defined function which is not contained in the complexity class PP-OBDD. We consider the following function.

Let $n = 2^\ell$. Define $\mathrm{SIP}_n \colon \{0,1\}^{2n+\ell} \to \{0,1\}$ ("shifted inner product") on vectors of variables $x = (x_0, \ldots, x_{n-1}) \in \{0,1\}^n$, $y = (y_0, \ldots, y_{n-1}) \in \{0,1\}^n$, and $s = (s_0, \ldots, s_{\ell-1}) \in \{0,1\}^\ell$ by

$$\mathrm{SIP}^i_n(x,y) := 1 \quad :\Leftrightarrow \quad \sum_{j=0}^{n-1} x_j y_{(i+j) \bmod n} \not\equiv 0 \bmod 2, \quad \text{for } i = 0, \ldots, n-1;$$

$$\mathrm{SIP}_n(x,y,s) := \bigvee_{0 \leq i \leq n-1} [|s|_2 = i] \ \wedge\ \mathrm{SIP}^i_n(x,y).$$

To prove the required lower bound for SIP_n, we apply the technique from Chapter 11.7 in the book. W. l. o. g. let n be even. Let X be the variable set of SIP_n and let $\pi \colon \{1, \ldots, 2n+\ell\} \to X$ describe a variable order for SIP_n. Let (X_1, X_2) be the partition with $X_1 := \{\pi(1), \ldots, \pi(k)\}$ and $X_2 = X - X_1$ where k is chosen such that X_1 contains exactly $n/2$ x-variables. Call variables x_r and y_s *partners* for the function SIP^i_n if $r + i \equiv s \bmod n$. By the pigeonhole principle it can be shown that there is an $i_0 \in \{0, \ldots, n-1\}$ such that for at least $m = n/2$ pairs of variables which are partners with respect to $\mathrm{SIP}_n^{i_0}$, both variables lie in different parts of the partition (X_1, X_2). It follows that there is a rectangle reduction from IP_m to $\mathrm{SIP}_n^{i_0}$ with respect to the partition (X_1, X_2). Hence, lower bounds on communication complexity for IP_m can immediately be translated into lower bounds on the OBDD size of SIP_n. By the result of Forster, we especially get that each randomized OBDD for SIP_n requires size at least $2^{n/4}$. It is straightforward to extend these ideas to kOBDDs (see also [3]), kIBDDs, and oblivious BPs of linear length by the standard techniques.

[1] J. Forster. A linear lower bound on the unbounded error probabilistic communication complexity. In *Proc. of 16th IEEE Int. Conf. on Computational Complexity*, 100–106, 2001.

[2] J. Forster, M. Krause, S. V. Lokam, R. Mubarakzjanov, N. Schmitt, H.-U. Simon. Relations between communication complexity, linear arrangements, and computational complexity. In *Proc. of 21st FST & TCS*, 171–182, 2001.

[3] M. Sauerhoff. On the size of randomized OBDDs and read-once branching programs for k-stable functions. *Computational Complexity* 10:155–174, 2001.

Open Problem 11.25. The equation is not true. The function weighted sum WS is contained in NP-FBDD and in coNP-FBDD but not in BPP-FBDD (and, therefore, neither in RP-FBDD nor in coRP-FBDD). This result has been obtained in the following paper. M. Sauerhoff: Randomness versus nondeterminism for read-once and read-k branching programs. STACS ’2003, LNCS 2607, 307-318, 2003.